Interaction of Waves of Current and Polarization*†

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The system considered here is a polarizable semiconductor through which a current is flowing in response to an externally applied electric field. Phenomenological equations relating the charge density, polarization, electric and magnetic fields, and atomic displacements in such a system are described. It is shown that a traveling wave of small amplitude oscillations of these quantities satisfies the equations when the frequency ω and propagation vector $\mathbf{q}$ satisfy a certain dispersion relation which is derived. For some choices of system parameters the dispersion relation admits solutions in which ω is real and $\mathbf{q}$ is complex with real and imaginary parts of opposite sign, suggesting the possibility that the system might support amplifying waves. Examples are given in which the parameters are as nearly as possible those appropriate to a crystal of indium antimonide. Some of the parameters which must be assigned depend on the drift velocity of the electrons, and it is difficult to determine appropriate values or ranges of values for them in the interesting region of large drift velocities.

I. INTRODUCTION

THE investigation described here was stimulated by the work of Hutson, McFee, and White,[1] who amplified ultrasonic waves in a piezoelectric semiconductor by applying a steady electric field in the direction of sound-wave propagation and increasing its magnitude until the drift velocity of the carriers exceeded the velocity of sound. Hutson and White,[2] White,[3] and Quate[4] have shown that the amplification depends on the transfer of energy from space-charge waves traveling on the drifting carriers to the sound waves. The work cited suggests the question: Is it possible to transfer energy from the space-charge waves on carriers drifting in a compound semiconductor to the polarization waves associated with the optical branches of the vibrational spectrum of such a crystal? In an attempt to answer that question we consider the phenomenological equations governing the behavior of a system consisting of carriers drifting in a polarizable medium under the influence of a steady electric field. Solutions of these equations in the form of decaying or possibly growing traveling waves are sought. The values of the various input parameters associated with these two types of solutions of the phenomenological equations are investigated. Finally, there is a preliminary discussion of the necessity for distinguishing between "amplifying" and "evanescent" growing waves.[5]

Coupled traveling waves of the type considered here, if they can be excited, might prove of considerable interest. The analysis given below indicates that they would have frequencies in the range around 10^{13} cps. They should have interesting and useful interactions with other crystal excitations and with electromagnetic radiation.

II. PHENOMENOLOGICAL EQUATIONS

Of the equations appearing in White's analysis,[3] the following four, involving the electric displacement $\mathbf{D}$, the carrier-current-density $\mathbf{J}$, and the electric field intensity $\boldsymbol{\mathcal{E}}$ are retained in the present work:

$$\nabla\cdot\mathbf{D}=-4\pi e n_s\,, \tag{1}$$

$$\nabla\cdot\mathbf{J}=e\partial n_s/\partial t\,, \tag{2}$$

$$\mathbf{J}=e\mu n_c\boldsymbol{\mathcal{E}}+e\mathfrak{D}_n\nabla n_c\,, \tag{3}$$

$$n_c=n_0+f n_s\,. \tag{4}$$

In these equations n_c is the density of carriers (here assumed to be electrons in the conduction band, in order to make the analysis more specific), n_0 is the undisturbed equilibrium value of n_c (which produces electrical neutrality when there is no wave present), n_s is the space-charge density expressed in units of electronic charge (a function of position and time), and f is the fraction of the space charge which is produced by mobile electrons (the remaining fraction of the space charge is produced by trapped electrons). The magnitude of the electronic charge is denoted by e, μ is the electron mobility, and $\mathfrak{D}_n$ is the electron diffusion constant. The first two equations should be valid in all cases. Equation (3) is expected to hold only if the frequency (τ^{-1}) of electron collisions is large compared to the frequency ($\omega/2\pi$) of the wave motion, i.e., $\omega\tau<1$, and if the wavelength ($2\pi/q$) of the wave motion is large compared to the mean free path l_e of the electrons, i.e., $ql_e<1$.

The description of the polarizable lattice is that developed by Born and Huang[6] for a diatomic crystal with optical isotropy. They introduce the vector $\mathbf{w}$, which is the displacement of the positive relative to

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† A brief account of this work was presented at the 1963 March meeting of the American Physical Society [Bull. Am. Phys. Soc. 8, 254 (1963)].

[1] A. R. Hutson, J. H. McFee, and D. L. White, Phys. Rev. Letters **7**, 237 (1961).

[2] A. R. Hutson and D. L. White, J. Appl. Phys. **33**, 40 (1962).

[3] D. L. White, J. Appl. Phys. **33**, 2547 (1962).

[4] C. F. Quate, M. L. Report No. 889, Microwave Laboratory, Stanford University (unpublished).

[5] P. A. Sturrock, Phys. Rev. **112**, 1488 (1958).

[6] M. Born and K. Huang, *Dynamical Theory of Crystal Lattices* (Oxford University Press, New York, 1954).

the negative ion multiplied by the square root of the reduced mass of the two ions per unit volume. Their equations involving **w** are

$$\partial^2\mathbf{w}/\partial t^2 = b_{11}\mathbf{w} + b_{12}\boldsymbol{\mathcal{E}}\,, \tag{5}$$

$$\mathbf{P} = b_{21}\mathbf{w} + b_{22}\boldsymbol{\mathcal{E}}\,, \tag{6}$$

where **P** is the dielectric polarization. The coefficients b_{ij} are real scalars which Born and Huang relate to ω_0, the infrared dispersion frequency, ϵ_0, the static dielectric constant, and ϵ_∞, the high-frequency dielectric constant as follows:

$$b_{11} = -\omega_0^2\,, \tag{7}$$

$$b_{12} = b_{21} = [(\epsilon_0 - \epsilon_\infty)/4\pi]^{1/2}\omega_0\,, \tag{8}$$

$$b_{22} = (\epsilon_\infty - 1)/4\pi\,. \tag{9}$$

For small carrier densities, the relations (7)–(9) are sufficiently well satisfied for purposes of the following argument. Equations (5) and (6) are valid "whenever conditions are everywhere practically uniform over regions containing many lattice cells,"[7] i.e., the analysis which follows is not valid for wavelengths much less than 100 Å.

The last three relations needed come from electromagnetic theory:

$$\mathbf{D} = \boldsymbol{\mathcal{E}} + 4\pi\mathbf{P}\,, \tag{10}$$

$$\nabla\times\boldsymbol{\mathcal{E}} = -c^{-1}(\partial\mathbf{H}/\partial t)\,, \tag{11}$$

$$\nabla\times\mathbf{H} = c^{-1}(4\pi\mathbf{J} + 4\pi\partial\mathbf{P}/\partial t + \partial\boldsymbol{\mathcal{E}}/\partial t)\,; \tag{12}$$

here **H** is the magnetic field and c is the velocity of light.

III. TRAVELING-WAVE SOLUTIONS OF THE EQUATIONS

What is sought are solutions of Eqs. (1)–(12) in which the components of the vectors **w**, $\boldsymbol{\mathcal{E}}$, **P**, **J**, **D**, and **H** and the scalar n_s are of the form

$$A = A_0 + A_1\exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)]\,,$$

with $A_1 \ll A_0$.

When the variables have this form, taking the curl of Eq. (11) and substituting Eq. (12) in the result in the usual way leads to the relations

$$\mathcal{E}_1{}^{\parallel} = (4\pi/i\omega)(J_1{}^{\parallel} - i\omega P_1{}^{\parallel}), \tag{13}$$

$$\boldsymbol{\mathcal{E}}_1{}^{\perp} = (4\pi i/\omega)(\omega/cq)^2[1-(\omega/cq)^2]^{-1}(\mathbf{J}_1{}^{\perp} - i\omega\mathbf{P}_1{}^{\perp}), \tag{14}$$

where $A_1{}^{\parallel}$ means the component of $\mathbf{A}_1$ parallel to **q** and $\mathbf{A}_1{}^{\perp}$ stands for a component of $\mathbf{A}_1$ perpendicular to **q**.

One such solution is obtained by reducing Eq. (5) to the form $\partial^2 w_1{}^{\parallel}/\partial t^2 = -\omega^2 w_1{}^{\parallel}$ with

$$\omega^2 = -b_{11}\{1 + K^2[1 + i\omega_c(\omega - fv_d q + if\mathfrak{D}_n q^2)^{-1}]^{-1}\}, \tag{15}$$

in which $K^2 = -4\pi b_{12}b_{21}/\epsilon_\infty b_{11} = (\epsilon_0 - \epsilon_\infty)/\epsilon_\infty$, $\omega_c = 4\pi\sigma/\epsilon_\infty$ (here $\sigma = n_0 e\mu$), and $v_d = -\mu\mathcal{E}_0{}^{\parallel}$. [This reduction can be performed by operating with $(\partial/\partial t)\nabla\cdot$ on Eq. (3), and substituting in the result the relation $\nabla\cdot\mathbf{J} = -(4\pi)^{-1}(\partial/\partial t)(\nabla\cdot\mathbf{D})$ which follows from Eqs. (1) and (2). The small second-order terms $(\nabla\cdot\mathbf{D})_1(\nabla\cdot\boldsymbol{\mathcal{E}})_1$ and $\boldsymbol{\mathcal{E}}_1\cdot\nabla(\nabla\cdot\mathbf{D})_1$ are neglected. The resulting relation between $D_1{}^{\parallel}$ and $\mathcal{E}_1{}^{\parallel}$ combined with Eq. (10) yields a relation between $\mathcal{E}_1{}^{\parallel}$ and $P_1{}^{\parallel}$ which is substituted in Eq. (6) to obtain $\mathcal{E}_1{}^{\parallel}$ in terms of $w_1{}^{\parallel}$. The elimination of $\mathcal{E}_1{}^{\parallel}$ between this last expression and Eq. (5) gives the indicated result.] Equation (15), the dispersion relation, is quadratic in q with the solutions

$$q = (\omega_D/2ifv_d)\{1 \pm [1 + 4(\omega_c/\omega_D) \times(\omega^2-\omega_0^2)/(\omega^2 - L\omega_0^2) - 4i(\omega/\omega_D)]^{1/2}\}, \tag{16}$$

in which $\omega_D = fv_d^2/\mathfrak{D}_n$, $L \equiv 1 + K^2$, and we have replaced b_{11} by its value $-\omega_0^2$ [Eq. (7)]. The principal features of these two solutions of the dispersion relation are discussed in the next section. The remainder of this section is devoted to obtaining another solution of Eqs. (1)–(12) associated with transverse waves.

A differential equation involving $\mathbf{w}_1{}^{\perp}$ and $w_1{}^{\parallel}$ can be obtained by the following operations:

The expression for n_c in terms of n_0 and $\nabla\cdot\mathbf{D}$ coming from Eqs. (1) and (4) is substituted in Eq. (3). Neglecting second-order terms, the transverse component of the resulting equation is

$$\mathbf{J}_1{}^{\perp} = n_0 e\mu\boldsymbol{\mathcal{E}}_1{}^{\perp} - (4\pi)^{-1}iqD_1{}^{\parallel}\mu f\boldsymbol{\mathcal{E}}_0{}^{\perp}. \tag{17}$$

In this equation we substitute the relation between $D_1{}^{\parallel}$ and $\mathcal{E}_1{}^{\parallel}$ referred to after Eq. (15), and then insert the resulting expression for $\mathbf{J}_1{}^{\perp}$ in Eq. (14), which becomes

$$\mathbf{P}_1{}^{\perp} = (4\pi)^{-1}[(cq/\omega)^2 - 1 - 4\pi i(\sigma/\omega)]\boldsymbol{\mathcal{E}}_1{}^{\perp} + i(\sigma/\omega)(q/\omega)\mathcal{E}_1{}^{\parallel}[1 + (q/\omega)\mu f\mathcal{E}_0{}^{\parallel} + i\omega(q/\omega)^2 f\mathfrak{D}_n]^{-1}\mu f\boldsymbol{\mathcal{E}}_0{}^{\perp}. \tag{18}$$

$\mathbf{P}_1{}^{\perp}$ from this equation is substituted in the transverse component of Eq. (6), which is then solved for $\boldsymbol{\mathcal{E}}_1{}^{\perp}$ in terms of $\mathbf{w}_1{}^{\perp}$ and $\boldsymbol{\mathcal{E}}_0{}^{\perp}$. (The coefficient of the term in $\boldsymbol{\mathcal{E}}_0{}^{\perp}$ involves $\mathcal{E}_1{}^{\parallel}$.) This expression for $\boldsymbol{\mathcal{E}}_1{}^{\perp}$ is inserted in the transverse component of Eq. (5), and $\mathcal{E}_1{}^{\parallel}$ is replaced by a complicated coefficient times $w_1{}^{\parallel}$, using the relation between $\mathcal{E}_1{}^{\parallel}$ and $w_1{}^{\parallel}$ referred to after Eq. (15). The final result is

$$\partial^2\mathbf{w}_1{}^{\perp}/\partial t^2 = b_{11}\left[1 + \frac{4\pi b_{12}b_{21}/\epsilon_\infty b_{11}}{\epsilon_\infty{}^{-1}(cq/\omega)^2 - 1 - i(\omega_c/\omega)}\right]\mathbf{w}_1{}^{\perp} + \frac{4\pi i(b_{12}b_{21}/\epsilon_\infty)(\omega_c/\omega)(q/\omega)\mu f w_1{}^{\parallel}}{[\epsilon_\infty{}^{-1}(cq/\omega)^2 - 1 - i(\omega_c/\omega)][1 + (q/\omega)\mu f\mathcal{E}_0{}^{\parallel} + i\omega(q/\omega)^2 f\mathfrak{D}_n + i(\omega_c/\omega)]}\boldsymbol{\mathcal{E}}_0{}^{\perp}. \tag{19}$$

[7] Reference 6, p. 83.

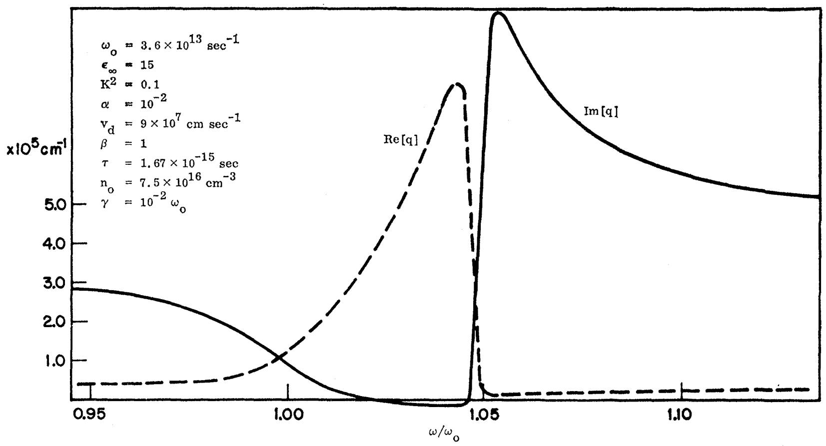

FIG. 1. Real and imaginary parts of the propagation constant, q, versus frequency, ω, for the indicated values of the parameters.

For $\mathcal{E}_0^{\perp}=0$, this equation describes the transverse vibrations of the lattice coupled to the electromagnetic field in the presence of carriers. When the carrier density vanishes ($\omega_c=0$), it reduces to the dispersion relation for the "optical-waves" in a polar insulator.[8] The term in $\mathcal{E}_0^{\perp}$ shows that it is possible to transfer energy to the carrier-damped "optical-wave" mode from the longitudinal mode previously discussed [Eq. (16)] if this latter mode can be excited in the presence of a component of steady electric field normal to its direction of propagation.

IV. DISCUSSION OF THE DISPERSION RELATION

The most interesting feature of the (complex) solutions (16) of the dispersion relation between q and ω is the presence of the "resonant" term $4(\omega_c/\omega_D)(\omega^2-\omega_0^2)/(\omega^2-L\omega_0^2)$ which becomes infinite as ω^2 approaches $L\omega_0^2$. It will now be shown that for appropriate values of the parameters this term causes the real and imaginary parts of q to be of opposite sign, implying a growing wave solution. It is desirable to modify the resonant term in Eq. (16) to take account of the fact that the lattice waves in any real crystal are damped. Born and Huang[9] have shown that the effects of damping are included well enough for many purposes by replacing ω_0^2 by $\omega_0^2-i\gamma\omega$, where γ is a damping coefficient. With this replacement, Eq. (16) becomes

$$q=\frac{\omega_D}{2ifv_d}\left\{1\pm\left[1+4\frac{\omega_c}{\omega_D}\frac{\omega^2-\omega_0^2+i\gamma\omega}{\omega^2-L(\omega_0^2-i\gamma\omega)}-4i\frac{\omega}{\omega_D}\right]^{1/2}\right\}. \quad (20)$$

Equation (20) leads to the following resolution of q into real and imaginary parts:

$$q=(\omega_D/2fv_d)[\pm G-i(1\pm F)], \quad (21)$$

with

$$F=[(1+\xi)^2+\eta^2]^{1/2}\cos(\theta/2), \quad (22)$$

$$G=[(1+\xi)^2+\eta^2]^{1/2}\sin(\theta/2), \quad (23)$$

$$\theta=\tan^{-1}[-\eta/(1+\xi)], \quad (-\pi\leqslant\theta\leqslant 0) \quad (24)$$

and

$$\xi=4(\omega_c/\omega_D)[(\omega^2-\omega_0^2)(\omega^2-L\omega_0^2)+L(\gamma\omega)^2] \div[(\omega^2-L\omega_0^2)^2+(L\gamma\omega)^2], \quad (25)$$

$$\eta=4(\omega/\omega_D)+4(\omega_c/\omega_D)K^2\gamma\omega^3 \div[(\omega^2-L\omega_0^2)^2+(L\gamma\omega)^2]. \quad (26)$$

To go any further in exploring the form and significance of the relation between ω and q, we must now assign values to the parameters involved in it. Available results of theory and experiment appear insufficient to determine reasonable values or possible ranges of values for $\mathfrak{D}_n$. For the purpose of obtaining a preliminary orientation as to the possible forms the solutions (21) of the dispersion relation might take, we make the unrealistic assumptions that $\mathfrak{D}_n$ will have the same value for electrons drifting at the high velocity v_d as for electrons drifting at very low velocities, and that frequencies which enter here are still sufficiently small and the wavelengths sufficiently large that Eq. (3) is always valid. Then relations valid for low drift velocities, low frequencies, and long wavelengths, such as the Einstein relation between diffusion and mobility, can be made to provide estimates of possible values for

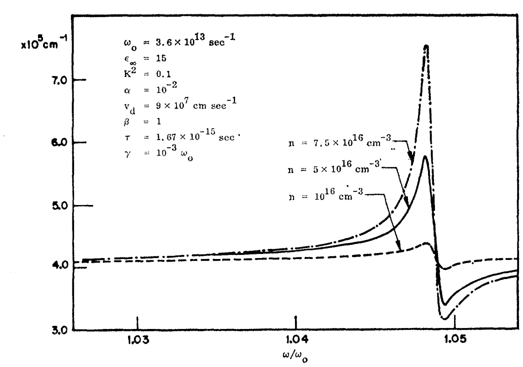

FIG. 2. Real part of the propagation constant, q versus frequency, ω, for three different carrier concentrations and the indicated values of the remaining parameters. Both abscissa and ordinate scales are linear.

[8] Reference 6, p. 94, Eq. (8.23).
[9] Reference 6, pp. 120–121.

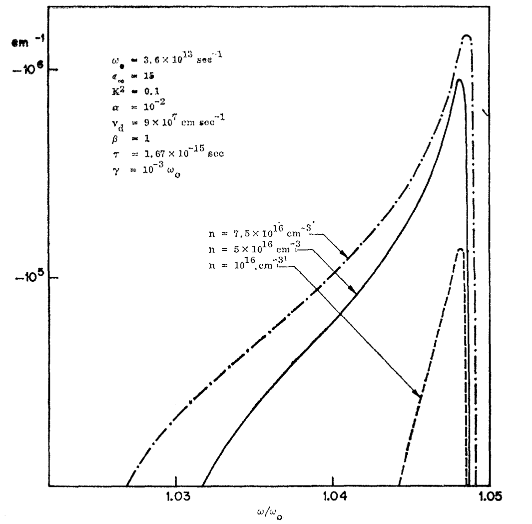

FIG. 3. Imaginary part of the propagation constant, q, versus frequency, ω, for three different carrier concentrations (the same as in Fig. 2) and the indicated values of the remaining parameters (also as in Fig. 2). Here the ordinate scale is logarithmic. Note also that negative values of $\text{Im}(q)$ are plotted upward.

these parameters. The Einstein relation is

$$\mu K_B T_n = e\mathfrak{D}_n\,, \tag{27}$$

with

$$\mu = e\tau/m^*; \tag{28}$$

K_B is Boltzmann's constant, and T_n is the temperature of the electron distribution, which, for low drift velocities, equals that of the lattice. These expressions imply

$$\omega_D = fm^*v_d^2/\tau K_B T_n = f\beta\tau^{-1} \tag{29}$$

with

$$\beta \equiv m^*v_d^2/K_B T_n\,. \tag{30}$$

Now the dispersion relation (21) depends on the following parameters:

$$\omega_0,\ \epsilon_\infty,\ K^2,\ \alpha\equiv m^*/m,\ v_d,\ \beta,\ \tau,\ n_0, \text{ and } \gamma.$$

In Figs. 1–3 we have plotted curves of the real and imaginary parts of q versus ω, as given by the branch of Eq. (21) associated with the lower signs, for a few choices of values of these parameters. Because the largest values of v_d reported[10] are for electrons in InSb, and since large values of v_d appear to favor the existence of growing wave solutions, we have chosen parameter values appropriate to electrons in InSb, wherever possible. Since reasonable values for τ at high v_d and for γ at low temperatures are unknown, the form of the dispersion curves for several widely varying choices of these parameters was investigated and it was found that the qualitative features of the curves given in Figs. 1–3 are unaltered by variations of τ and γ over one or more decades around the values assigned them in these figures.

[10] M. Glicksman and W. A. Hicinbothem, Jr., Phys. Rev. **129**, 1572 (1963).

The figures show that for values of ω in a range around ω_0 (which range may be broad or narrow, depending on the values of the other parameters), the real and imaginary parts of q may differ in sign. With this sign difference, the approximate solution of the system of equations discussed here, in which all quantities vary as $\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$, corresponds to a wave growing in amplitude as it propagates. A demonstration such as the preceding one that a particular system supports growing monochromatic waves is not sufficient to prove that it can support useful amplifying wave packets.[5] Sturrock[5] has described a method for determining from the dispersion relation for the system whether its "growing" waves can be superposed to form "amplifying" wave packets or whether they are merely "evanescent" waves. This type of investigation of the dispersion relation for the system considered in this paper has not yet been completed. However, the fact that the growing waves discussed by White[3] could be shown experimentally[1] to be amplifying encourages the hope that the growing waves in the somewhat similar system treated here may also be amplifying rather than evanescent.

It should be noted that the interesting and experimentally realizable ranges of parameters appear to be seriously limited by the allowable power dissipation. Relations valid for small drift velocities lead to this expression for the average rate of power loss per unit volume from a steady stream of electrons:

$$P_0 = n_0 m^* v_d^2/\tau = f^{-1} n_0 K_B T_n \omega_D\,. \tag{31}$$

Inspection of Eqs. (21)–(26) suggests that in order for growing wave solutions to exist, ω_D must be larger than a threshold value of about $4\,L^{1/2}\omega_0$. Then to keep the power dissipation within tolerable limits, even for microsecond pulses, it is necessary to have $10^{-6}\,n_0\omega_D < 10^{22}$ ($\approx$number of atoms/cm³), or $n_0 < 10^{28}\omega_D^{-1}$ or $n_0 \lesssim 10^{14}$. To obtain growing waves with this limit on n_0, it may be necessary to make $\beta \gg 1$ (so that τ and, hence, ω_c can be made large) and/or $\gamma \ll \omega_0$.

It is clear that we need more knowledge of the diffusive behavior of electrons drifting at high velocities, as well as of the collision processes which limit drift velocities and of the damping of optical-mode vibrations at low temperatures, in order to determine whether or not the parameter ranges associated with growing waves are accessible to experiment.

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